Outline for Static Games of Complete Information

- I. Definition of a game
- II. Examples
- III. Definition of Nash equilibrium
- IV. Examples, continued
- V. Iterated elimination of dominated strategies
- VI. Mixed-strategy Nash equilibria
- VII. Correlated equilibria
- VIII. Existence theorem on Nash equilibria
- IX. The Hotelling model and extensions

Definition: An *n*-player, **static game** of complete information consists of an *n*-tuple of strategy sets and an *n*-tuple of payoff functions, denoted by $G = \{S_1, \dots , S_n; u_1, \dots , u_n\}$

S_i, the **strategy set** of player *i*, is the set of all permissible moves for player *i*. We write $s_i \in S_i$ for one of player *i*'s strategies.

u_i, the **payoff function** of player *i*, is the utility, profit, etc. for player *i*, and depends on the strategies chosen by all the players: $u_i(s_1, \dots , s_n)$.

Example: Prisoners’ Dilemma

		Prisoner II	
		Remain Silent	Confess
Prisoner I	Remain Silent	–1 , –1	–5 , 0
	Confess	0 , –5	–4 ,–4

Example: Battle of the Sexes

		F	
		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

Definition: A **Nash equilibrium** of G (in pure strategies) consists of a strategy for every player with the property that no player can improve her payoff by unilaterally deviating:

$(s_1^*, \ldots , s_n^*)$ with the property that, for every player i:

$$u_i\,(s_1^*, \ldots , s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots , s_n^*)$$
$$\geq\ u_i\,(s_1^*, \ldots , s_{i-1}^*, s_i, s_{i+1}^*, \ldots , s_n^*)$$

for all $s_i \in S_i$.

Equivalently, a Nash equilibrium is a mutual best response. That is, for every player i, s_i^* is a solution to:

$$s_i^* \in \arg\max_{s_i \in S_i} \left\{ u_i(s_1^*, \ldots , s_{i-1}^*, s_i, s_{i+1}^*, \ldots , s_n^*) \right\}$$

Example: Prisoners' Dilemma

		Prisoner II	
		Remain Silent	Confess
Prisoner I	Remain Silent	–1 , –1	–5 , 0
	Confess	0 , –5	–4 , –4

Example: Battle of the Sexes

		F	
		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

Cournot (1838) Model of Oligopoly

(a) n firms
(b) Each firm i has a constant marginal (and average) cost of c_i
(c) Inverse aggregate demand function of $P(Q)$
(d) Each firm simultaneously and independently selects a strategy consisting of a ***quantity*** $q_i \in [0, a]$ (where $P(a) = 0$)

Then, with two firms, the payoff functions are:

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - c_1 q_1$$

$$\pi_2(q_1, q_2) = q_2 P(q_1 + q_2) - c_2 q_2 \text{ .}$$

and the strategy sets are:

$$S_1 = [0, a] \qquad S_2 = [0, a]$$

It is often also convenient to assume a common marginal cost (i.e., $c_1 = c = c_2$) and a linear demand curve $P(Q) = a - Q$.

Solution of Cournot Model with Two Firms

(q_1^*, q_2^*) is a Nash equilibrium if and only if:

$$q_1^* \text{ solves } \max_{q_1}\{q_1\,[P(q_1 + q_2^*) - c]\}$$

and $$q_2^* \text{ solves } \max_{q_2}\{q_2\,[P(q_1^* + q_2) - c]\}.$$

With $P(Q) = a - Q$, we get first order conditions:

$$q_1(-1) + a - q_1 - q_2^* - c\Big|_{q_1 = q_1^*} = 0$$

$$\Rightarrow \quad (1)\;\; a - 2q_1^* - q_2^* = c$$

and: $$q_2(-1) + a - q_1^* - q_2 - c\Big|_{q_2 = q_2^*} = 0$$

$$\Rightarrow \quad (2)\;\; a - q_1^* - 2q_2^* = c$$

Subtracting (1) – (2) gives:

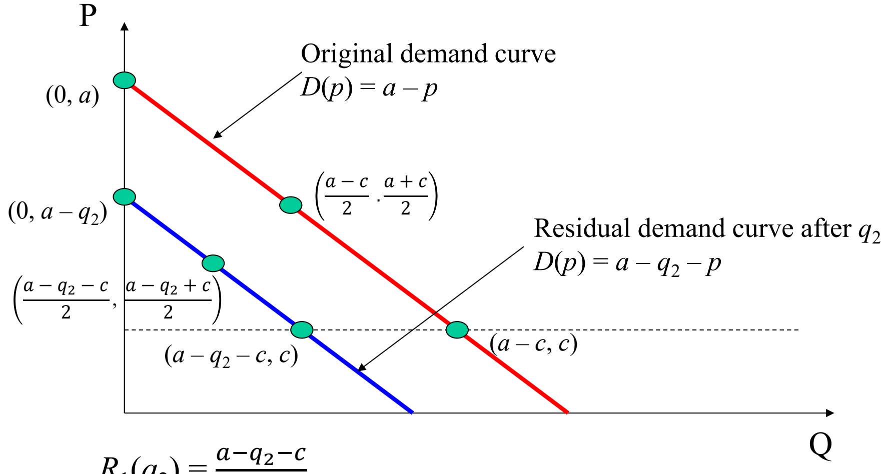
$$q_2^* - q_1^* = 0$$

Substituting $q_2^* = q_1^*$ into (1) gives:

$$a - 2q_1^* - q_1^* = c$$

$$\boldsymbol{q_1^* = \frac{a-c}{3}\;;\;\; q_2^* = \frac{a-c}{3}}\;\; .$$

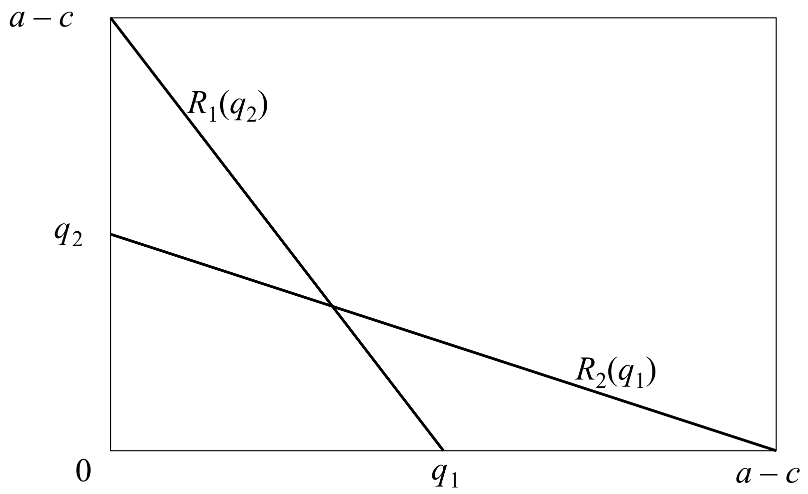
Best Response for Firm 1 to q_2



$$R_1(q_2) = \frac{a-q_2-c}{2}$$

Similarly, the best response for firm 2 to q_1 is:

$$R_2(q_1) = \frac{a-q_1-c}{2}$$



Cournot Duopoly: Best Response Functions

Bertrand (1883) Model of Oligopoly

(a) n firms
(b) Each firm i has a constant marginal (and average) cost of c_i
(c) Aggregate demand function of $Q(P)$
(d) Each firm simultaneously and independently selects a strategy consisting of a ***price*** $p_i \in [0, a]$ (where $Q(a) = 0$)

Then, with two firms, the payoff functions are:

$$\pi_1(p_1, p_2) = \begin{cases} Q(p_1)[p_1 - c_1], & \text{if } p_1 < p_2 \\ \tfrac{1}{2}Q(p_1)[p_1 - c_1], & \text{if } p_1 = p_2 \\ 0, & \text{if } p_1 > p_2 \end{cases}$$

and

$$\pi_2(p_1, p_2) = \begin{cases} Q(p_2)[p_2 - c_2], & \text{if } p_2 < p_1 \\ \tfrac{1}{2}Q(p_2)[p_2 - c_2], & \text{if } p_2 = p_1 \\ 0, & \text{if } p_2 > p_1 \end{cases}$$

Bertrand (1883) Model of Oligopoly

As in the Cournot game, the strategy sets are:

$S_1 = [0, a] \qquad S_2 = [0, a]$

and it is again usually convenient to assume a common marginal cost (i.e., $c_1 = c = c_2$).

The unique solution of the Bertrand game with two firms and common marginal cost $c_1 = c = c_2$ is as follows:

Bertrand (1883) Model of Oligopoly

Observation 1: In any Nash equilibrium (p_1^*, p_2^*), it must be the case that $p_1^* \geq c$ and $p_2^* \geq c$.

Proof: Suppose otherwise. Without loss of generality, say $p_1^* \leq p_2^*$ and $p_1^* < c$. Then firm 1 is currently earning strictly negative profits and could profitably deviate to $p_1^* \geq c$ (thereby instead earning nonnegative profits).

Bertrand (1883) Model of Oligopoly

Observation 2: In any Nash equilibrium (p_1^*, p_2^*), it must be the case that $p_1^* = p_2^*$.

Proof: Suppose otherwise. Without loss of generality, say $p_1^* < p_2^*$ (and $p_1^* \geq c$). Then firm 2 is currently earning zero profits and, if $p_1^* > c$, firm 2 can profitably deviate to $p_2^* = p_1^* - \varepsilon$. Meanwhile, if $p_1^* = c$, firm 1 can profitably deviate to $p_1^* = p_2^* - \varepsilon$.

Bertrand (1883) Model of Oligopoly

Observation 3: The unique Nash equilibrium is (p_1^*, p_2^*) = (c, c).

Proof: By Observations 1 and 2, the only remaining possibility is $p_1^* = p^* = p_2^* > c$. Then each firm is currently earning profits of: $\tfrac{1}{2}Q(p^*)[p^* - c]$

and either firm could profitably deviate to $p^* - \varepsilon$ and thereby come arbitrarily close to earning: $Q(p^*)[p^* - c]$.

Q.E.D.

The Pollution Game

Consumers have a choice of three different models of cars, which are identical in all respects except for price and emissions:

Model A: p_A = \$25,000; e_A = 100 units
Model B: p_B = \$26,000; e_B = 10 units
Model C: p_C = \$27,000; e_C = 0 units

A consumer's utility from using a car is given by:

$$U = v - p - E$$

where v = reservation value of a car;
p = price paid for model bought;
$E = \sum_{i=1}^{N} e_i$ = aggregate emissions (over all consumers)
where e_i = 100 or 10 or 0 , depending on which model is purchased by consumer i.

For any number of players, N, satisfying $11 < N < 100$, the societal optimal choice is for every player to purchase Model B. Calculate:

U_i if every player purchases Model A =
$= (v - 25{,}000 - E)$
$= (v - 25{,}000 - 100N)$

U_i if every player purchases Model B =
$= (v - 26{,}000 - E)$
$= (v - 26{,}000 - 10N)$

U_i if every player purchases Model C =
$= (v - 27{,}000 - E)$
$= (v - 27{,}000)$

For example:

U_i if every player purchases Model A – U_i if every player purchases Model B =
$= (v - 25{,}000 - 100N) - (v - 26{,}000 - 10N) < 0$
$\Rightarrow 1{,}000 - 90N < 0 \Rightarrow N > 11$

However, let E_{-i} denote the total emissions from all of player i's opponents. Then:

U_i from Model A – U_i from Model B =

$$= (v - 25{,}000 - E_{-i} - 100) - (v - 26{,}000 - E_{-i} - 10)$$
$$= 1{,}000 - 90$$
$$= 910$$

and:

U_i from Model B – U_i from Model C =

$$= (v - 26{,}000 - E_{-i} - 10) - (v - 27{,}000 - E_{-i} - 0)$$
$$= 1{,}000 - 10$$
$$= 990$$

In conclusion, irrespective of the choices made by the other players, player *i* gets a higher payoff from Model A than from Model B, and player *i* gets a higher payoff from Model B than from Model C.

Dominated strategies:

Strategy s_i (strictly) **dominates** strategy s_i' if, for ***all*** possible strategy combinations of opponents, s_i yields a (strictly) higher payoff than s_i' to player *i*.

Iterated elimination of strictly dominated strategies:

Eliminate all strategies that are strictly dominated, relative to opponents' strategies that have not yet been eliminated.

A few more notes on the Pollution Game.

This is a classic example of an ***externality***: a situation where one player's action enters directly into another player's payoff function.

Broadly speaking, externalities can be addressed with either ***standards*** or ***taxes***.

The appropriate standard in this model: it is only legal to sell Model B (or better).

How do you calculate the appropriate tax?

The ***private cost*** of Model A (instead of Model B) = 100 – 10 = 90

The ***social cost*** of Model A (instead of Model B) = $(100 - 10) \times N = 90N$

The appropriate tax is the difference between the social cost and the private cost, here $90(N - 1)$.

This is the amount that leads the decision maker to ***internalize the externality***.

Approximately the same outcome can be reached with a ***tax*** on Model A or with a ***subsidy*** on Model B. However, there may be general equilibrium effects.

Dominated strategies:

Strategy s_i (strictly) **dominates** strategy s_i' if, for ***all*** possible strategy combinations of opponents, s_i yields a (strictly) higher payoff than s_i' to player i.

Iterated elimination of strictly dominated strategies:

Eliminate all strategies that are strictly dominated, relative to opponents' strategies that have not yet been eliminated.

Example: Prisoners’ Dilemma

		Prisoner II	
		Remain Silent	Confess
Prisoner I	Remain Silent	–1 , –1	–5 , 0
	Confess	0 , –5	–4 , –4

		Player II	
		Left	**Right**
Player I	**Top**	~~**1 , 2**~~	~~**4 , 1**~~
	Middle	**3 , 2**	~~**2 , 1**~~
	Bottom	~~**2 , 1**~~	~~**1 , 3**~~

Bottom is strictly dominated by Middle (for Player I)

Right is strictly dominated by Left (for Player II)

Top is strictly dominated by Middle (for Player I)

Results on Iterated Elimination of Strictly Dominated Strategies

Proposition 1: If iterated elimination of strictly dominated strategies yields a ***unique*** strategy n-tuple, then this strategy n-tuple is the ***unique*** Nash equilibrium (and it is a ***strict*** Nash equilibrium). [See Gibbons text, pp. 12 – 14.]

> (Definition: A ***strict*** Nash equilibrium is a strategy n-tuple with the property that every unilateral deviation makes the deviator ***strictly*** worse off.)

Proposition 2: Every Nash equilibrium survives iterated elimination of strictly dominated strategies.

Proposition 3: Iterated elimination of strictly dominated strategies is order-independent.

Proposition 2: Every Nash equilibrium survives iterated elimination of strictly dominated strategies.

Proof: Suppose not. Then there is a Nash equilibrium $s^* = (s_1^*, \dots, s_i^*, \dots, s_n^*)$ that gets eliminated. Without loss of generality, assume that s_i^* is the ***first*** component of s^* that is eliminated. Let us say that when s_i^* is eliminated, it is eliminated by s_i'. Then $u_i(s_1, \dots, s_i', \dots, s_n) > u_i(s_1, \dots, s_i^*, \dots, s_n)$ for each $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ that can be constructed from strategies that have not yet been eliminated. In particular, since s_i^* was assumed to be the first component of s^* to be eliminated, we can select $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$ for the opponents' strategies. This implies that:

$$u_i(s_1^*, \dots, s_i', \dots, s_n^*) > u_i(s_1^*, \dots, s_i^*, \dots, s_n^*),$$

i.e., s_i' is a profitable unilateral deviation for i against $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$, contradicting our hypothesis that s^* is a Nash equilibrium. ■

Guess 2/3 of the Average

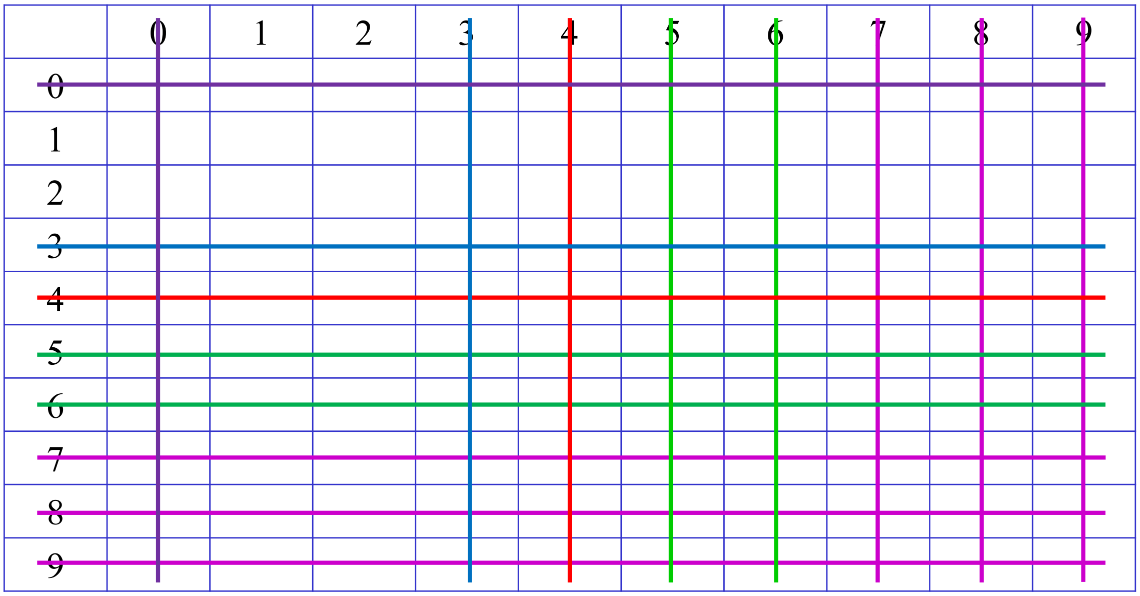
The Problem

Each of you have to choose an integer between 0 and 9, with the objective of guessing "2/3 of the average of the responses given by all students in the course".

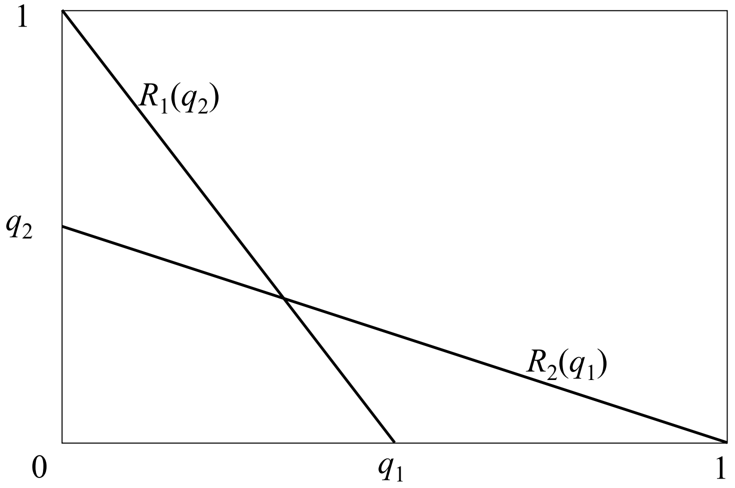
Each student who guesses an integer which is closest to 2/3 of the average of all of the responses rounded up to the nearest integer, wins. The winners equally divide a prize.

What is your guess?

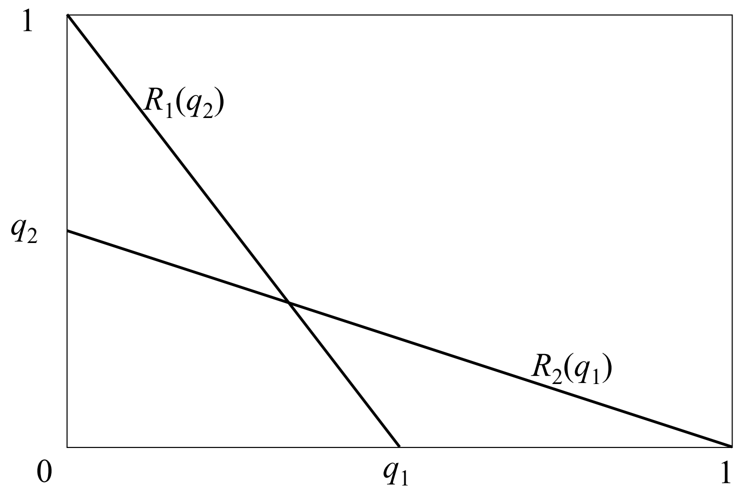
	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										



	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										



Cournot Duopoly: Best Response Functions



$q_1 > ½$ is strictly dominated by $q_1 = ½$

$q_2 > ½$ is strictly dominated by $q_2 = ½$

$q_1 < ¼$ is strictly dominated by $q_1 = ¼$

$q_2 < ¼$ is strictly dominated by $q_2 = ¼$

Example: Matching Pennies

II

		Heads	Tails
I	Heads	1 , –1	–1 , 1
	Tails	–1 , 1	1 , –1

Definition: Let player *i* have *K* pure strategies available. Then a **mixed strategy** for player *i* is a probability distribution over those *K* strategies.

Notation:

Strategy set:

$$S_i = \{s_{i1}, \ldots , s_{iK}\}$$

Mixed strategy:

$$p_i = (p_{i1}, \ldots , p_{iK})$$

such that $\sum_{k=1}^{K} p_{ik} = 1$

and each p_{ik} is between zero and one ($0 \leq p_{ik} \leq 1$).

Facts:

1. Theorem (Nash, 1950):
   Every finite game has at least one Nash equilibrium (when mixed strategies are permitted).

2. If, in a mixed-strategy Nash equilibrium, player *i* places positive probability on each of two strategies, then player *i* must be indifferent between these two strategies (i.e., they yield player *i* the same expected payoff).

		II	
		Heads	Tails
I	Heads	1 , –1	–1 , 1
	Tails	–1 , 1	1 , –1

Let q denote the probability with which Player I plays H, and let r denote the probability with which Player II plays H. We will solve for the NE by determining the value of r that makes Player I indifferent between H and T, and the value of q that makes Player II indifferent between H and T.

$$\mathrm{E}U_{\mathrm{I}}(\mathrm{H}) = r(1) + (1 - r)(-1) = 2r - 1.$$

$$\mathrm{E}U_{\mathrm{I}}(\mathrm{T}) = r(-1) + (1 - r)(1) = 1 - 2r.$$

Player I is indifferent between H and T if and only if:

$$\mathrm{E}U_{\mathrm{I}}(\mathrm{H}) = \mathrm{E}U_{\mathrm{I}}(\mathrm{T}) \Leftrightarrow 2r - 1 = 1 - 2r \Leftrightarrow r = \tfrac{1}{2}\,.$$

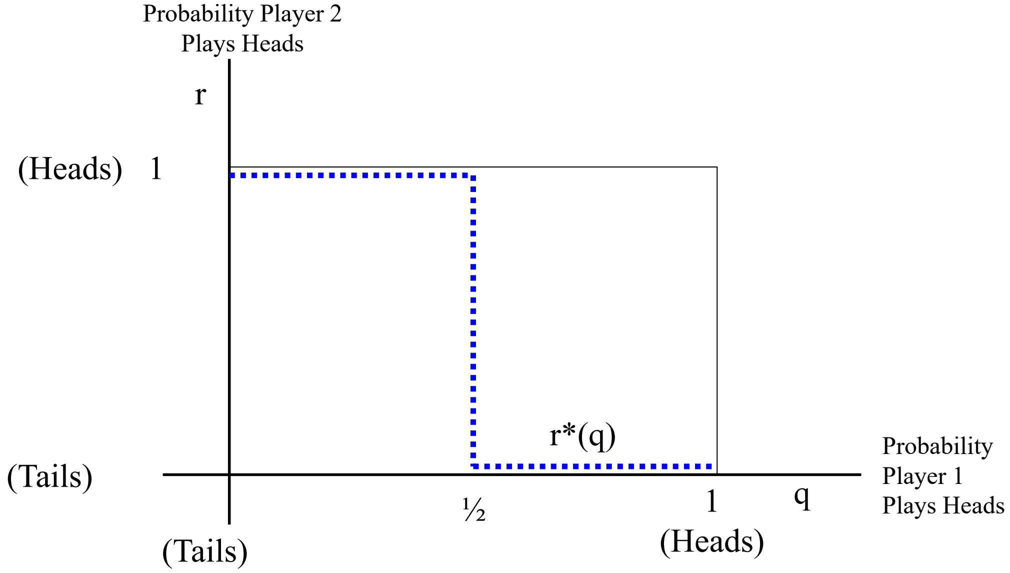
Similarly:

$$\mathrm{E}U_{\mathrm{II}}(\mathrm{H}) = q(-1) + (1 - q)(1) = 1 - 2q.$$

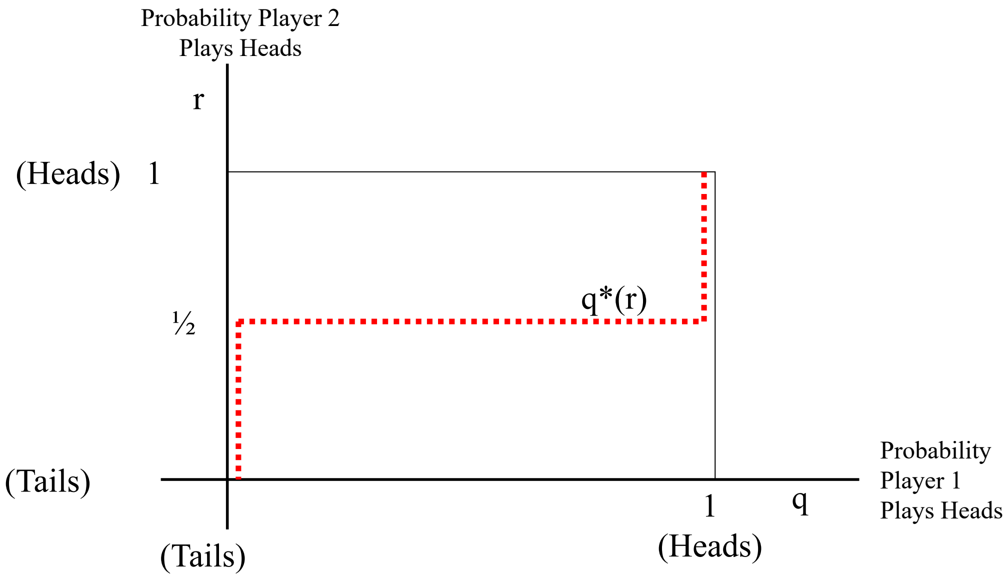
$$\mathrm{E}U_{\mathrm{II}}(\mathrm{T}) = q(1) + (1 - q)(-1) = 2q - 1.$$

Player II is indifferent between H and T if and only if:

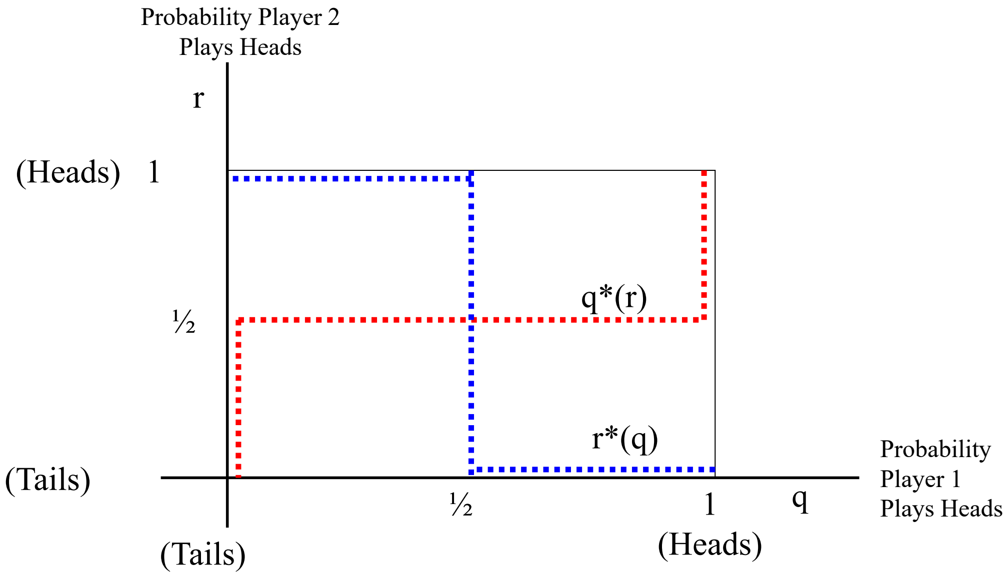
$$\mathrm{E}U_{\mathrm{II}}(\mathrm{H}) = \mathrm{E}U_{\mathrm{II}}(\mathrm{T}) \Leftrightarrow 1 - 2q = 2q - 1 \Leftrightarrow q = \tfrac{1}{2}\,.$$



Best response correspondence of Player 2



Best response correspondence of Player 1



Matching Pennies

Example: Battle of the Sexes

F

		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

F

		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

Let q denote the probability with which M plays Boxing, and let r denote the probability with which F plays Boxing. We will solve for the NE by determining the value of r that makes M indifferent between Boxing and Ballet, and the value of q that makes F indifferent between Boxing and Ballet.

$$\mathrm{E}U_{\mathrm{M}}(\text{Boxing}) = r(2) + (1 - r)(0) = 2r.$$

$$\mathrm{E}U_{\mathrm{M}}(\text{Ballet}) \;= r(0) + (1 - r)(1) = 1 - r.$$

M is indifferent between Boxing and Ballet if and only if:

$$\mathrm{E}U_{\mathrm{M}}(\text{Boxing}) = \mathrm{E}U_{\mathrm{M}}(\text{Ballet}) \Leftrightarrow 2r = 1 - r \Leftrightarrow r = 1/3 \;.$$

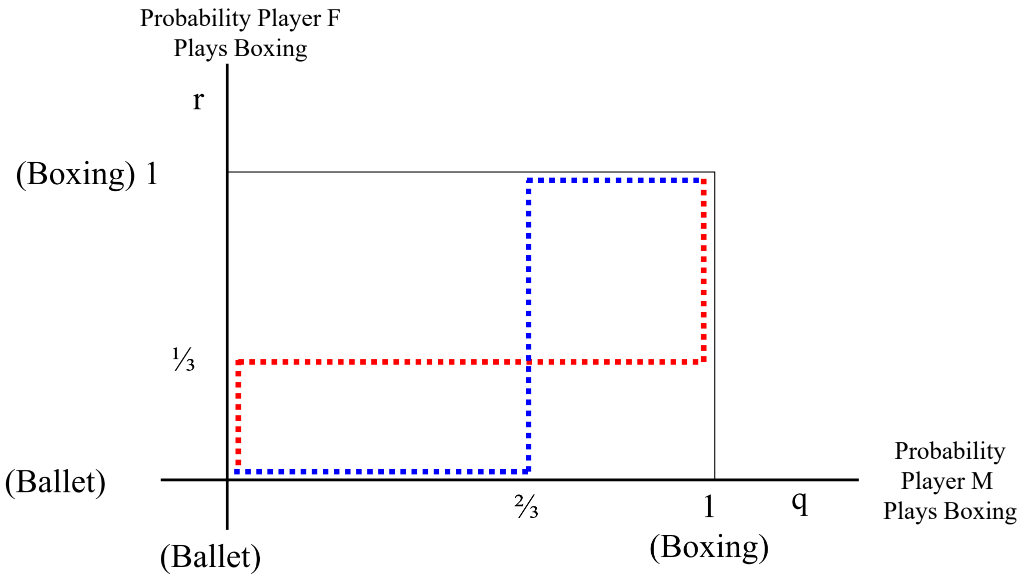
Similarly:

$$\mathrm{E}U_{\mathrm{F}}(\text{Boxing}) = q(1) + (1 - q)(0) = q.$$

$$\mathrm{E}U_{\mathrm{F}}(\text{Ballet}) \;= q(0) + (1 - q)(2) = 2 - 2q.$$

F is indifferent between Boxing and Ballet if and only if:

$$\mathrm{E}U_{\mathrm{F}}(\text{Boxing}) = \mathrm{E}U_{\mathrm{F}}(\text{Ballet}) \Leftrightarrow q = 2 - 2q \Leftrightarrow q = 2/3 \;.$$



Battle of the Sexes

Correlated Equilibrium I: Public Randomizing Device

Suppose that there exists a public randomizing device that comes up "heads" ½ the time and "tails" ½ the time.
Then the players could agree to play {Boxing, Boxing} when "heads" and {Ballet, Ballet} when "tails".
Example: Play {Boxing, Boxing} when the closing DJIA is an even number and play {Ballet, Ballet} when it is an odd number; achieving E payoffs of (3/2, 3/2) — better than the mixed strategy Nash equilibrium.

		F	
		Boxing	Ballet
M	Boxing	2 , 1	0 , 0
	Ballet	0 , 0	1 , 2

Correlated Equilibrium II: Mediated communication

Consider the following game:

		II	
		L	R
I	T	5 , 1	0 , 0
	B	4 , 4	1 , 5

A public randomizing device of ½ – ½ enables us to obtain expected payoffs of (3, 3).

However, a mediator could randomize among three instructions — (T,L), (B,L) and (B,R). The mediator tells player I whether to play T or B (but ***not*** what he has told player II). Similarly, he tells player II whether to play L or R (but ***not*** what he has told player I).

It can be shown: (i) if Prob(T,L) = 1/3, Prob(B,R) = 1/3, and Prob(B,L) = 1/3, then no player has any incentive to deviate from these instructions; and (ii) E payoffs are now (10/3, 10/3)!

Correlated Equilibrium II: Mediated communication

In greater detail:

		II	
		L	R
I	T	1/2	0
	B	0	1/2

This is clearly incentive compatible and enables the players to obtain expected payoffs of (3,3).

Bayes' Rule

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

A, B = events
P(A|B) = probability of A given B is true
P(B|A) = probability of B given A is true
P(A), P(B) = the independent probabilities of A and B

Correlated Equilibrium II: Mediated communication

In greater detail:

		II	
		L	R
I	T	1/3	0
	B	1/3	1/3

By Bayes' law, Prob (II is told R | I is told B) = $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$.

Meanwhile, Prob (II is told R | I is told T) = 0.

Hence, player I's E payoff from playing B when told to play B is ½ (4) + ½ (1) = 2 ½ .
Player I's E payoff from playing T when told to play B is also ½ (5) + ½ (0) = 2 ½ .

Thus: (i) it is incentive compatible for Player I to follow the mediator's instructions (and symmetrically for Player II); and (ii) E payoffs are now (10/3, 10/3)!

Correlated Equilibrium II: Mediated communication

Derivation (limiting attention to ***symmetric*** correlated equilibria):

			II
		L	R
I	T	½(1 – p)	0
	B	p	½(1 – p)

By Bayes' law, Prob (II is told R | I is told B) = $\frac{\frac{1}{2}(1-p)}{p+\frac{1}{2}(1-p)}$.

By Bayes' law, Prob (II is told L | I is told B) = $\frac{p}{p+\frac{1}{2}(1-p)}$.

Optimal correlated equilibrium:

$$\max_{p\in[0,1]}\left\{p\cdot 4+\tfrac{1}{2}(1-p)\cdot 5+\tfrac{1}{2}(1-p)\cdot 1\right\}$$

$$\text{s.t. }\; p\cdot 4+\tfrac{1}{2}(1-p)\cdot 1\geq p\cdot 5+\tfrac{1}{2}(1-p)\cdot 0$$

Results on correlated equilibrium:

1. With attention limited to public randomizing devices, the set of outcomes of correlated equilibria is the convex combination of all (pure- and mixed-strategy) Nash equilibrium outcomes.

2. With mediated communication possible, one can sometimes construct correlated equilibria that outperform any convex combination of Nash equilibria—see, for example, the previous slides.

Nash Existence Theorem (Nash, 1950):

Every finite game has at least one Nash equilibrium (when mixed strategies are permitted).

Definition of **finite game**:

- Finitely many players; and
- Each player's strategy set, S_i, is finite.

Common fixed point theorems in economics:

Banach Fixed Point Theorem (contraction mapping theorem):

Suppose that (X, d) is a complete metric space. Also suppose that the function $f: X \to X$ is a ***contraction mapping***, i.e., $d(f(x), f(y)) \leq \delta d(x,y)$ for $\delta < 1$. Then there exists a ***unique fixed point*** of f, i.e., a point $x \in X$ such that $x = f(x)$.

Tarski Fixed Point Theorem:

Suppose that $(X, \leq)$ is a complete lattice. Also suppose that the function $f: X \to X$ is monotonic with respect to $\leq$. Then there exists a ***fixed point*** of f, i.e., a point $x \in X$ such that $x = f(x)$, and the set of fixed points of f in X also forms a complete lattice under $\leq$.

This and next class (Nash existence theorem):

Brouwer Fixed Point Theorem:

Suppose that X is a nonempty, compact, convex set in $\mathbb{R}^n$. Also suppose that the ***function*** $f : X \rightarrow X$ is continuous. Then there exists a ***fixed point*** of f, i.e., a point $x \in X$ such that $x = f(x)$.

Kakutani Fixed Point Theorem:

Suppose X as above. Also suppose that the ***correspondence*** $F : X \rightarrow X$ is nonempty and convex-valued, and that $F(\cdot)$ has a closed graph. Then there exists a ***fixed point*** of F, i.e., a point $x \in X$ such that $x \in F(x)$.

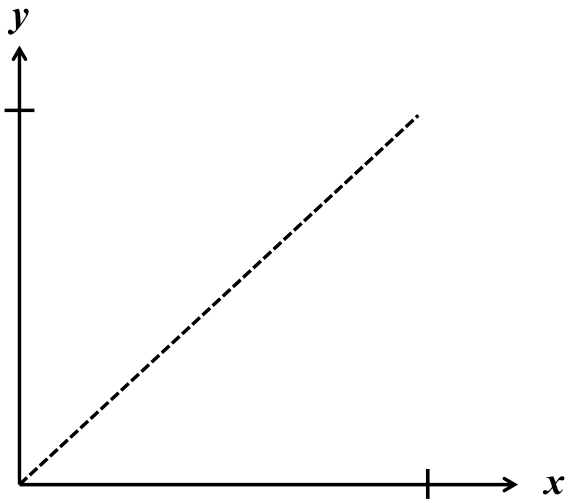
Brouwer Fixed Point Thm:

X

- nonempty
- compact
  - closed
  - bounded
- convex
- $\subset \mathbb{R}^n$

f

- function
- maps $X \rightarrow X$
- continuous



Brouwer Fixed Point Theorem:

Suppose that X is a nonempty, compact, convex set in $\mathbb{R}^n$. Also suppose that the ***function*** $f : X \to X$ is continuous. Then there exists a ***fixed point*** of f, i.e., a point $x \in X$ such that $x = f(x)$.

Kakutani Fixed Point Theorem:

Suppose X as above. Also suppose that the ***correspondence*** $F : X \to X$ is nonempty and convex-valued, and that $F(\cdot)$ has a closed graph. Then there exists a ***fixed point*** of F, i.e., a point $x \in X$ such that $x \in F(x)$.

Notes:

(1) The correspondence $F(\cdot)$ is said to have a ***closed graph*** if, simply, the graph of $F(\cdot)$ is a closed set. That is, $F(\cdot)$ has a closed graph if it has the property that whenever the sequence $(x^n, y^n) \to (x, y)$, with $y^n \in F(x^n)$ for every n, then $y \in F(x)$.

Essentially the same as upper hemicontinuity (u.h.c.).

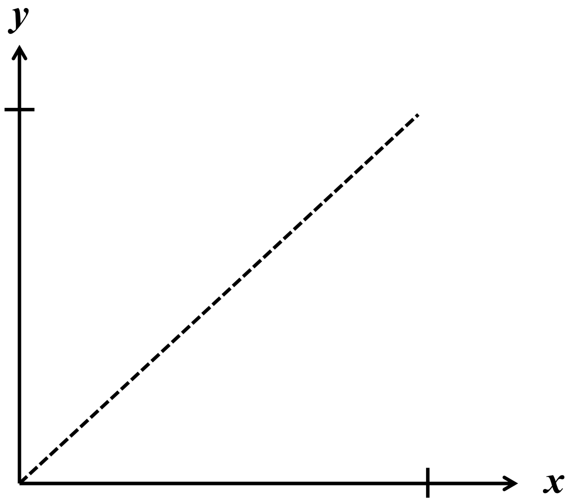
Kakutani Fixed Point Thm:

X

- nonempty
- compact
  - closed
  - bounded
- convex
- $\subset \mathbb{R}^n$

F

- correspondence
- maps $X \to X$
- nonempty- and convex-valued
- closed graph



Nash Existence Theorem **(Nash, 1950):**

Every finite game has at least one Nash equilibrium (when mixed strategies are permitted).

Notation:

$$\vec{\boldsymbol{p}} = \begin{pmatrix}\vec{p}_1\\ \vdots \\ \vec{p}_i \\ \vdots \\ \vec{p}_n\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}p_{11}\\ \vdots\\ p_{1K_1}\end{pmatrix}\\ \vdots \\ \begin{pmatrix}p_{i1}\\ \vdots\\ p_{iK_i}\end{pmatrix}\\ \vdots \\ \begin{pmatrix}p_{n1}\\ \vdots\\ p_{nK_n}\end{pmatrix}\end{pmatrix}$$

The correspondence *F*:

$$F:\begin{pmatrix}\vec{p}_1\\ \vdots\\ \vec{p}_i\\ \vdots\\ \vec{p}_n\end{pmatrix}\to\begin{pmatrix}BR_1(\vec{\boldsymbol{p}}_{-1})\\ \vdots\\ BR_i(\vec{\boldsymbol{p}}_{-i})\\ \\ BR_n(\vec{\boldsymbol{p}}_{-n})\end{pmatrix}$$

Fixed point:

$$\vec{\boldsymbol{p}} \in F(\vec{\boldsymbol{p}})$$
implies
$$\vec{p}_i \in BR_i(\vec{\boldsymbol{p}}_{-i}) \text{ for all } i$$

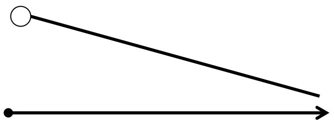
We need:

X
- nonempty
- compact
  - closed
  - bounded
- convex
- $\subset \mathbb{R}^n$

F
- correspondence
- maps $X \to X$
- nonempty- and convex-valued
- closed graph

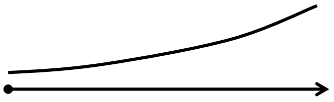
Nonempty-valued: For best response to exist, we need a maximum to exist

- Continuous function on compact set has a maximum; hence, we require:
  - – closed



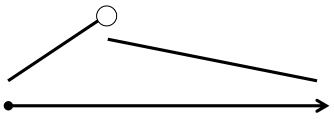
or there may be no max

  - – bounded



or there may be no max

  - – continuous



or there may be no max

<u>Notes</u>:

(1) The correspondence $F(\cdot)$ is said to have a ***closed graph*** if, simply, the graph of $F(\cdot)$ is a closed set. That is, $F(\cdot)$ has a closed graph if it has the property that whenever the sequence $(x^n, y^n) \to (x, y)$, with $y^n \in F(x^n)$ for every n, then $y \in F(x)$.

Essentially the same as upper hemicontinuity (u.h.c.).

(2) The best-response correspondence $BR_i(\cdot)$ of each player i has a closed graph, by the following argument.

Suppose that there is a sequence $(x^n, y^n) \to (x, y)$ such that $y^n \in BR_i(x^n)$ for every n, but $y \notin BR_i(x)$. Then there exists $\varepsilon > 0$ and $y' \neq y$ such that:

$$u_i(y', x) \ > \ u_i(y, x) + \varepsilon \, .$$

But this contradicts:

$$u_i(y', x^n) \ \leq \ u_i(y^n, x^n), \text{ for every } n.$$

Example of using Brouwer Fixed Point Thm to prove existence of equilibrium.

Consider an n-firm Cournot game with fairly general demand curves and marginal costs:

$$\pi_i(q_i, q_{-i}) = [P(q_i + \Sigma q_{-i}) - c_i]\, q_i \, ,$$

making the assumption that $\arg\max_{q_i} \{[P(q_i + \Sigma q_{-i}) - c_i]\, q_i\}$ is always single-valued.

Then we can define $X = [0, a]^n$ and use the following f:

$$f : \begin{pmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{pmatrix} \to \begin{pmatrix} BR_1(\bar{q}_{-1}) \\ \vdots \\ BR_i(\bar{q}_{-i}) \\ \vdots \\ BR_n(\bar{q}_{-n}) \end{pmatrix}$$

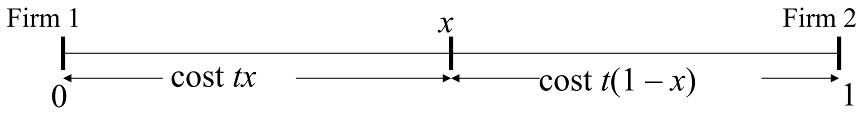
The Brouwer Fixed Point Theorem guarantees that f has a fixed point.

Product Differentiation: The Hotelling Model

Consumers are uniformly distributed on the interval [0, 1].

There are two firms, located at $x = 0$ and $x = 1$, which each produce the same physical good at marginal cost of c.

Consumers have transportation cost t per unit of distance.



Each consumer consumes 0 or 1 units of the good:

$u(0) = 0$; $u(1) = v$.

If firm 1 charges p_1 and firm 2 charges p_2, the consumer located at x gets $v - p_1 - tx$ from purchasing at firm 1 and gets $v - p_2 - t(1 - x)$ from purchasing at firm 2.

Let $\tilde{x}$ denote the customer who is indifferent between purchasing at firm 1 and firm 2. Then:

$$v - p_1 - t\tilde{x} = v - p_2 - t(1 - \tilde{x})$$

$$2t\tilde{x} = t + p_2 - p_1$$

$$\tilde{x} = \tfrac{1}{2} + \tfrac{p_2 - p_1}{2t}.$$

The profits of firm 1 are given by:

$$\pi_1(p_1, p_2) = [p_1 - c]\,\tilde{x} = [p_1 - c][\tfrac{1}{2} + \frac{p_2 - p_1}{2t}].$$

The profits of firm 2 are given by:

$$\pi_2(p_1, p_2) = [p_2 - c][1 - \tilde{x}] = [p_2 - c][\tfrac{1}{2} - \frac{p_2 - p_1}{2t}].$$

These imply the first-order conditions of:

(1) $c + t + p_2^* - 2p_1^* = 0$

(2) $c + t + p_1^* - 2p_2^* = 0$.

Solving yields:

$\boldsymbol{p_1^* = c + t}$; $\boldsymbol{p_2^* = c + t}$ **.**